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**Circulant Preconditioners for
Hermitian Toeplitz Systems**

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Circulant Preconditioners for Hermitian Toeplitz Systems

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Abstract. We study the solutions of Hermitian positive definite Toeplitz systems $Ax = b$ by the preconditioned conjugate gradient method for three families of circulant preconditioners C . The convergence rates of these iterative methods depend on the spectrum of $C^{-1}A$. For a Toeplitz matrix A with entries which are Fourier coefficients of a positive function f in the Wiener class, we establish the invertibility of C , and that the spectrum of the preconditioned matrix $C^{-1}A$ clusters around one. We prove that if f is $(l+1)$ -times differentiable, with $l > 0$, then the error after $2q$ conjugate gradient steps has decreased like $((q-1)!)^{-2l}$. We also show that if C copies the central diagonals of A , then C minimizes $\|C - A\|_1$ and $\|C - A\|_\infty$.

Abbreviated Title. Hermitian Toeplitz Systems

Key words. Toeplitz matrix, circulant matrix, preconditioned conjugate gradient method

AMS(MOS) subject classifications. 65F10, 65F15

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1. Introduction.

In this paper we discuss the solutions to a class of Hermitian positive definite Toeplitz systems $Ax = b$ by the preconditioned conjugate gradient method. Direct methods that are based on the Levinson recursion formula are in constant use; see for instance, Levinson [10] and Trench [12]. For an n by n Toeplitz matrix A_n , these methods require $O(n^2)$ operations. Faster algorithms that require $O(n \log^2 n)$ operations have been developed, see Bitmead and Anderson [1] and Brent, Gustavson and Yun [2]. The stability properties of these direct methods for symmetric positive definite matrices are discussed in Bunch [3].

In [11], Strang proposed using preconditioned conjugate gradient method with circulant preconditioners for solving symmetric positive definite Toeplitz systems. The number of operations per iteration is of order $O(n \log n)$ as circulant systems can be solved efficiently by the Fast Fourier Transform. R. Chan and Strang [4] then considered using a circulant preconditioner C_n that is obtained by copying the central diagonals of A_n and bringing them around to complete the circulant. In that paper, we proved that if the underlying generating function f , the Fourier coefficients of which give the entries of A_n , is a positive function in the Wiener class, then for n sufficiently large, C_n and C_n^{-1} are uniformly bounded in the l_2 norm and that the eigenvalues of the preconditioned matrix $C_n^{-1}A_n$ cluster around 1. We note that f is an even function as A_n are symmetric.

In this paper, we extend these results to Hermitian positive definite Toeplitz systems. More precisely, we show in §2 that if the generating function f is a real-valued positive function in the Wiener class, then the spectrum of $C_n^{-1}A_n$ is clustered around 1. We remark that the proof given in R. Chan and Strang [4] cannot be readily generalized to cover this case. In fact, for Hermitian A_n , the Hankel matrices $H_{n/2}$ used in the proof in [4] are not Hermitian, and the Circulant-Toeplitz eigenvalue problem cannot be split into two similar Toeplitz-Hankel eigenvalue problems. In §3, we establish the superlinear convergence rate of the conjugate gradient method when applied to these preconditioned systems. In particular, we show that if f is $(l+1)$ -times differentiable, with $l > 0$, then the error after $2q$ conjugate gradient steps has decreased like $((q-1)!)^{-2l}$.

In §4, we discuss other viable preconditioners for the same problem. We show that the preconditioned systems for these preconditioners also have clustered spectra around 1 for large n and that they all have the same asymptotic convergence rate. In §5, we show that the preconditioner that copies the central diagonals of A_n is optimal in the sense that it minimizes $\|F_n - A_n\|_1 = \|F_n - A_n\|_\infty$ over all Hermitian circulant matrices F_n . Finally, numerical results are given in §6.

2. The Spectrum of the Preconditioned Matrix.

Let us first assume that the Hermitian Toeplitz matrices A_n are finite sections of a fixed singly infinite positive definite matrix A_∞ , see R. Chan and Strang [4]. Thus the (i, j) -th entries of A_n and A_∞ are a_{i-j} , with $a_k = \bar{a}_{-k}$ for all k . We associate with A_∞ the real-valued generating function

$$f(\theta) = \sum_{-\infty}^{\infty} a_k e^{-ik\theta},$$

defined on $[0, 2\pi)$. We will assume that f is a positive function and is in the Wiener class, i.e. the sequence $\{a_k\}_{k=-\infty}^{\infty}$ is in l_1 . It then easily follows that A_n are Hermitian positive definite matrices for all n , see for instance, Grenander-Szegő [8]. Moreover, if

$$0 < f_{\min} < f < f_{\max} < \infty,$$

then the spectrum $\sigma(A_n)$ of A_n satisfies

$$\sigma(A_n) \subseteq [f_{\min}, f_{\max}]. \quad (1)$$

Let C_n be the Hermitian circulant preconditioner that copies the central diagonals of A_n . More precisely, the entries $c_{ij} = c_{i-j}$ of C_n are given by

$$c_k = \begin{cases} a_k & 0 \leq k \leq m, \\ a_{k-n} & m < k < n, \\ \bar{a}_{-k} & 0 < -k < n. \end{cases} \quad (2)$$

For simplicity, we are assuming here and in the following that $n = 2m + 1$. The case where $n = 2m$ can be treated similarly, and in that case, we define $c_m = (a_m + a_{-m})/2$, see (17) below.

We will show that $C_n^{-1}A_n$ has a clustered spectrum. We first note that

Theorem 1. *Suppose f is positive and is in the Wiener class. Then for large n , the circulants C_n and C_n^{-1} are uniformly bounded in the l_2 norm. In fact, for large n , the spectrum $\sigma(C_n)$ of C_n satisfies*

$$\sigma(C_n) \subseteq [f_{\min}, f_{\max}]. \quad (3)$$

The proof of this Theorem is similar to the proof of Theorem 1 of R. Chan and Strang [4], and we therefore omit it.

Next we show that $A_n - C_n$ has a clustered spectrum:

Theorem 2. *Let f be a positive function in the Wiener class, then for all $\epsilon > 0$, there exist M and $N > 0$ such that for all $n > N$, at most M eigenvalues of $C_n - A_n$ have absolute values exceeding ϵ .*

Proof: Clearly $R_n = C_n - A_n$ is a Hermitian Toeplitz matrix with entries $r_{ij} = r_{i-j}$ given by

$$r_k = \begin{cases} 0 & 0 \leq k \leq m, \\ a_{k-n} - a_k & m < k < n, \\ \bar{r}_{-k} & 0 < -k < n. \end{cases} \quad (4)$$

Since f is in the Wiener class, thus for all given $\epsilon > 0$, there exists an $N > 0$, such that $\sum_{k=N+1}^{\infty} |a_k| < \epsilon$. Let $U_n^{(N)}$ be the n by n matrix obtained from R_n by replacing the $(n - N)$ by $(n - N)$ principal submatrix of R_n by the zero matrix. Then $\text{rank}(U_n^{(N)}) \leq 2N$. Let $W_n^{(N)} \equiv R_n - U_n^{(N)}$. The leading $(n - N)$ by $(n - N)$ block of $W_n^{(N)}$ is the $(n - N)$ by $(n - N)$ principal submatrix of R_n , hence it is Toeplitz, and it is easy to see that the maximum column sum of $W_n^{(N)}$ is attained at the first column (or the $(n - N - 1)$ -th column). Thus

$$\|W_n^{(N)}\|_1 = \sum_{k=m+1}^{n-N-1} |r_k| = \sum_{k=m+1}^{n-N-1} |a_{k-n} - a_k| \leq \sum_{k=N+1}^{n-N-1} |a_k| < \epsilon. \quad (5)$$

Since $W_n^{(N)}$ is Hermitian, we have $\|W_n^{(N)}\|_{\infty} = \|W_n^{(N)}\|_1$. Thus

$$\|W_n^{(N)}\|_2 \leq (\|W_n^{(N)}\|_1 \cdot \|W_n^{(N)}\|_{\infty})^{\frac{1}{2}} < \epsilon.$$

Hence the spectrum of $W_n^{(N)}$ lies in $(-\epsilon, \epsilon)$. By Cauchy Interlace Theorem, see Wilkinson [13], we see that at most $2N$ eigenvalues of $R_n = C_n - A_n$ have absolute values exceeding ϵ . $\square$

Combining Theorems 1 and 2, and using the fact that

$$C_n^{-1}A_n = I_n + C_n^{-1}(A_n - C_n),$$

we have

Corollary. Let f be a positive function in the Wiener class, then for all $\epsilon > 0$, there exist N and $M > 0$, such that for all $n > N$, at most M eigenvalues of $C_n^{-1}A_n - I_n$ have absolute values larger than ϵ .

Thus the spectrum of $C_n^{-1}A_n$ is clustered around one for large n .

3. Superlinear Convergence Rate

It follows easily from the Corollary of the last section that the conjugate gradient method, when applied to the preconditioned system $C_n^{-1}A_n$, converges superlinearly. More precisely, for all $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that the error vector e_q at the q -th iteration satisfies

$$\|e_q\| \leq C(\epsilon)\epsilon^q\|e_0\|, \tag{6}$$

where $\|x\|^2 \equiv x^*C_n^{-\frac{1}{2}}AC_n^{-\frac{1}{2}}x$, see R. Chan and Strang [4] for a proof. Thus the number of iterations to achieve a fixed accuracy remains bounded as the matrix order n is increased. Since each iteration requires $O(n \log n)$ operations using the Fast Fourier Transform, see Strang [11], the work of solving the equation $A_n x = b$ to a given accuracy δ is $c(f, \delta)n \log n$, where $c(f, \delta)$ is a constant that depends on f and δ only.

We note that if extra smoothness conditions are imposed on f , we can get a more precise bound on the convergence rate:

Theorem 3. *Let f be a $(l+1)$ -times differentiable function with its $(l+1)$ -th derivative of f in $L^1[0, 2\pi)$, $l > 0$. Then for large n ,*

$$\|e_{2q}\| \leq \frac{c^q}{((q-1)!)^{2l}} \|e_0\|, \quad (7)$$

for some constant c that depends on f and l only.

Proof: We remark that from the standard error analysis of the conjugate gradient method, we have

$$\|e_q\| \leq \left[\min_{P_q} \max_{\lambda} |P_q(\lambda)| \right] \|e_0\|, \quad (8)$$

where the minimum is taken over polynomials of degree q with constant term 1 and the maximum is taken over the spectrum of $C_n^{-1}A_n$, or equivalently, the spectrum of $C_n^{-\frac{1}{2}}A_nC_n^{-\frac{1}{2}}$, see for instance, Golub and van Loan [7]. In the following, we will try to estimate that minimum.

We first note that the assumptions on f imply that

$$|a_j| \leq \frac{\hat{c}}{|j|^{l+1}} \quad \forall j,$$

where $\hat{c} = \|f^{(l+1)}\|_{L^1}$, see, for instance, Katznelson [9]. Hence

$$\sum_{j=k+1}^{n-k-1} |a_j| \leq \hat{c} \sum_{j=k+1}^{n-k-1} \frac{1}{|j|^{l+1}} \leq \hat{c} \int_k^\infty \frac{dx}{x^{l+1}} \leq \frac{\hat{c}}{k^l}, \quad \forall k \geq 1. \quad (9)$$

As in Theorem 2, we write

$$R_n = W_n^{(k)} + U_n^{(k)}, \quad \forall k \geq 1,$$

where $U_n^{(k)}$ is the matrix obtained from R_n by replacing its $(n-k)$ by $(n-k)$ principal submatrix of R_n by a zero matrix. Using the arguments in Theorem 2, cf (5) and (9), we see that $\text{rank}(U_n^{(k)}) \leq 2k$ and $\|W_n^{(k)}\|_2 \leq \hat{c}/k^l$, for all $k \geq 1$. Now consider

$$C_n^{-\frac{1}{2}}R_nC_n^{-\frac{1}{2}} = C_n^{-\frac{1}{2}}W_n^{(k)}C_n^{-\frac{1}{2}} + C_n^{-\frac{1}{2}}U_n^{(k)}C_n^{-\frac{1}{2}} \equiv \tilde{W}_n^{(k)} + \tilde{U}_n^{(k)}.$$

By Theorem 1, we have $\text{rank}(\tilde{U}_n^{(k)}) \leq 2k$ for large n and

$$\|\tilde{W}_n^{(k)}\|_2 \leq \|C_n^{-1}\|_2 \|W_n^{(k)}\|_2 \leq \frac{\tilde{c}}{k^l}, \quad \forall k \geq 1, \quad (10)$$

with $\tilde{c} = \hat{c}/f_{\min}$.

Next we note that $W_n^{(k)} - W_n^{(k+1)}$ can be written as the sum of two rank one matrices of the form:

$$W_n^{(k)} - W_n^{(k+1)} = u_k v_k^* + v_k u_k^* = \frac{1}{2}(w_k^+ w_k^{+\ast} - w_k^- w_k^{-\ast}), \quad \forall k \geq 0.$$

Here u_k is the $(n-k)$ -th unit vector, $v_k = (r_{n-k-1}, \dots, r_1, r_0/2, 0, \dots, 0)$, with r_j given by (4), and $w_k^\pm = u_k \pm v_k$. Hence by letting $z_k^\pm = C_n^{-\frac{1}{2}} w_k^\pm$ for $k \geq 0$, we have

$$\begin{aligned} C_n^{-\frac{1}{2}} R_n C_n^{-\frac{1}{2}} &= \tilde{W}_n^{(0)} = \tilde{W}_n^{(k)} + \frac{1}{2} \sum_{j=0}^{k-1} (z_j^+ z_j^{+\ast} - z_j^- z_j^{-\ast}), \\ &= \tilde{W}_n^{(k)} + V_k^+ - V_k^-, \quad \forall k \geq 1, \end{aligned} \quad (11)$$

where $V_k^\pm \equiv \frac{1}{2} \sum_{j=0}^{k-1} z_j^\pm z_j^{\pm\ast}$ are positive semi-definite matrices of rank k . Let us order the eigenvalues of $\tilde{W}_n^{(0)}$ as

$$\mu_0^- \leq \mu_1^- \leq \dots \leq \mu_1^+ \leq \mu_0^+.$$

By applying Cauchy Interlace Theorem to (11) and using the bound of $\|\tilde{W}_n^{(k)}\|_2$ in (10), we see that for all $k \geq 1$, there are at most k eigenvalues of $\tilde{W}_n^{(0)}$ lying to the right of $\tilde{c}/k^l$, and there are at most k of them lying to the left of $-\tilde{c}/k^l$. More precisely, we have

$$|\mu_k^\pm| \leq \|\tilde{W}_n^{(k)}\|_2 \leq \frac{\tilde{c}}{k^l}, \quad \forall k \geq 1.$$

Using the identity

$$C_n^{-\frac{1}{2}} A_n C_n^{-\frac{1}{2}} = I_n + C_n^{-\frac{1}{2}} R_n C_n^{-\frac{1}{2}} = I_n + \tilde{W}_n^{(0)},$$

we see that if we order the eigenvalues of $C_n^{-\frac{1}{2}} A_n C_n^{-\frac{1}{2}}$ as

$$\lambda_0^- \leq \lambda_1^- \leq \dots \leq \lambda_1^+ \leq \lambda_0^+,$$

then $\lambda_k^\pm = 1 + \mu_k^\pm$ for all $k \geq 0$ with

$$1 - \frac{\tilde{c}}{k^l} \leq \lambda_k^- \leq \lambda_k^+ \leq 1 + \frac{\tilde{c}}{k^l}, \quad \forall k \geq 1. \quad (12)$$

For $\lambda_0^\pm$, the bounds are obtained from (1) and (3):

$$\frac{f_{\min}}{f_{\max}} \leq \lambda_0^- \leq \lambda_0^+ \leq \frac{f_{\max}}{f_{\min}}. \quad (13)$$

Having obtained the bounds for $\lambda_k^\pm$, we can now construct the polynomial that will give us a bound for (8). Our idea is to choose P_{2q} that annihilates the q extreme pairs of eigenvalues. Thus consider

$$p_k(x) = \left(1 - \frac{x}{\lambda_k^+}\right)\left(1 - \frac{x}{\lambda_k^-}\right), \quad \forall k \geq 1.$$

Between those roots $\lambda_k^\pm$, the maximum of $|p_k(x)|$ is attained at the average $x = \frac{1}{2}(\lambda_k^+ + \lambda_k^-)$, where by (12), we have

$$\max_{x \in [\lambda_k^-, \lambda_k^+]} |p_k(x)| = \frac{(\lambda_k^+ - \lambda_k^-)^2}{4\lambda_k^+ \lambda_k^-} \leq \left(\frac{2\tilde{c}}{k^l}\right)^2 \cdot \left(\frac{f_{\max}}{2f_{\min}}\right)^2 = \left(\frac{\tilde{c}f_{\max}}{f_{\min}}\right)^2 \cdot \frac{1}{k^{2l}}, \quad \forall k \geq 1,$$

Similarly, for $k = 0$, we have, by using (13),

$$\max_{x \in [\lambda_0^-, \lambda_0^+]} |p_0(x)| = \frac{(\lambda_0^+ - \lambda_0^-)^2}{4\lambda_0^+ \lambda_0^-} \leq \frac{(f_{\max}^2 - f_{\min}^2)^2}{4f_{\min}^4}.$$

Hence the polynomial $P_{2q} = p_0 p_1 \cdots p_{q-1}$, which annihilates the q extreme pairs of eigenvalues, satisfies

$$|P_{2q}(x)| \leq \frac{c^q}{((q-1)!)^{2l}}, \quad (14)$$

for some constant c that depends only on f and l . This holds for all $\lambda_k^\pm$ in the inner interval between λ_{q-1}^- and λ_{q-1}^+ , where the remaining eigenvalues are. Equation (7) now follows directly from (8) and (14). $\square$

4. Other Circulant Preconditioners.

The proof of Theorem 2 suggests that there are many other viable preconditioners that can give us the same asymptotic convergence rate. One example is given by the circulant matrix B_n proposed by T. Chan [6]. It is obtained by averaging the corresponding diagonals of A_n with the diagonals of A_n being extended to length n by a wrap-around. More precisely, the entries $b_{ij} = b_{i-j}$ of B_n are given by

$$b_k = \begin{cases} \frac{1}{n} \{ka_{k-n} + (n-k)a_k\} & 0 \leq k < n, \\ \bar{b}_{-k} & 0 < -k < n, \end{cases}$$

where a_n is taken to be 0. He proved that such B_n minimizes the Frobenius norm $\|B_n - A_n\|_F$ over all possible circulant matrices B_n . The entries $r_{ij} = r_{i-j}$ of $B_n - A_n$ are given by

$$r_k = \begin{cases} \frac{k}{n}(a_{k-n} - a_k) & 0 \leq k < n, \\ \bar{r}_{-k} & 0 < -k < n. \end{cases}$$

As in Theorem 2, we let $W_n^{(N)}$ to be the matrix obtained from $B_n - A_n$ by replacing the last N rows and N columns of $B_n - A_n$ by zero vectors. We see that

$$\|W_n^{(N)}\|_1 \leq 2 \sum_{k=0}^{n-N-1} |r_k| \leq 2 \sum_{k=0}^N \frac{k}{n} |a_k| + 4 \sum_{k=N+1}^n |a_k|. \quad (15)$$

Now let $M > N$ be such that $\frac{1}{M} \sum_{k=0}^N k|a_k| < \epsilon$. Then for all $n > M$, we have $\|W_n^{(N)}\|_1 < 6\epsilon$. Hence the eigenvalues of $B_n - A_n$ are clustered around zero, except for at most $2N$ of them. We remark that by using results in R. Chan [5], we can show that $\lim_{n \rightarrow \infty} \|C_n - B_n\|_2 = 0$ and that the convergence rate of $C_n^{-1}A_n$ and $B_n^{-1}A_n$ are the same for large n . In particular, both will converge superlinearly.

As another example, let us consider the circulant matrix D_n with entries $d_{ij} = d_{i-j}$ given by

$$d_k = \begin{cases} a_{k-n} + a_k & 0 \leq k < n, \\ \bar{d}_{-k} & 0 < -k < n, \end{cases}$$

where a_n is again taken to be 0. The entries $r_{ij} = r_{i-j}$ of $D_n - A_n$ are given by

$$r_k = \begin{cases} a_{k-n} & 0 \leq k < n, \\ \bar{r}_{-k} & 0 < -k < n. \end{cases}$$

It is easily seen that the conclusion of Theorem 2 holds for this preconditioner too, cf (5) and (15). Similar to the case of B_n , we can also show that $\lim_{n \rightarrow \infty} \|C_n - D_n\|_2 = 0$ and that the convergence rate of $C_n^{-1}A_n$ and $D_n^{-1}A_n$ are the same for large n , see R. Chan [5]. Numerical results in §6 indeed show that the three preconditioners B_n , C_n and D_n behave almost the same for large n .

5. The Optimality of C_n .

From the discussion in §§2 and 4, we know that it is interesting to obtain the Hermitian circulant matrix F_n that minimizes the norm $\|F_n - A_n\|_1 = \|F_n - A_n\|_\infty$. The minimum is attained by C_n :

Theorem 4. *The circulant matrix C_n whose entries are given by (2) minimizes $\|F_n - A_n\|_1 = \|F_n - A_n\|_\infty$ over all possible Hermitian circulant matrices F_n .*

Proof: Let us construct the circulant matrix G_n that minimizes the column sums of $G_n - A_n$. Let the (i, j) -th entries of G_n be $g_{ij} = g_{i-j}$. Since G_n is Hermitian and circulant, we have $g_k = \bar{g}_{n-k}$ for $k = 1, \dots, m$, where $m = (n-1)/2$. Hence G_n is determined by $\{g_k\}_{k=0}^m$. For $j = 0, \dots, n-1$, the j -th column sum s_j of $G_n - A_n$ is given by

$$s_j = \sum_{k=0}^{n-1-j} |a_k - g_k| + \sum_{k=1}^j |\bar{a}_k - \bar{g}_k|. \quad (16)$$

We note that $s_{n-1-j} = s_j$ for $0 \leq j < n$. Hence it suffices to consider s_j for $0 \leq j \leq m$. The term involving g_0 in (16) is $|a_0 - g_0|$ which has its minimum at $g_0 = a_0$. For $k = 1, \dots, m$, the terms involving g_k in (16) are either of the form

$$\begin{aligned} \text{(a)} \quad & |a_k - g_k| + |\bar{a}_k - \bar{g}_k| = 2|a_k - g_k|, \\ \text{or (b)} \quad & |a_k - g_k| + |a_{n-k} - g_{n-k}| = |a_k - g_k| + |\bar{a}_{n-k} - \bar{g}_k|. \end{aligned}$$

In case (a), the minimum is at $g_k = a_k$. In case (b), the minimum occurs at any g_k lying on the line segment joining a_k and $\bar{a}_{n-k}$. In particular, (a) and (b) attain their minima at $g_k = a_k$. Thus G_n so constructed is the same as the C_n given by (2).

Now for any other circulant matrix E_n , the j -th column sum t_j of $E_n - A_n$ will satisfy $s_j \leq t_j$, for $j = 0, \dots, n-1$. Hence,

$$\|C_n - A_n\|_1 = \max_j s_j \leq \max_j t_j = \|E_n - A_n\|_1. \quad \square$$

Remark: When $n = 2m$ is even, g_m is real as G_n is both Hermitian and circulant. The term involving g_m in s_j takes the form $|a_m - g_m|$ or $|\bar{a}_m - g_m|$. Since $s_j = s_{n-1-j}$ for $j = 0, \dots, n-1$, we see that g_m should be chosen such that both terms are minimized, i.e.,

$$g_m = \frac{1}{2}(a_m + \bar{a}_m). \quad (17)$$

6. Numerical Results.

To test the convergence rates of the preconditioners, we have applied the preconditioned conjugate gradient method to $A_n x = b$ with

$$a_k = \begin{cases} \frac{1 + \sqrt{-1}}{(1+k)^{1.1}} & k > 0, \\ 2 & k = 0, \\ \bar{a}_{-k} & k < 0. \end{cases}$$

The underlying generating function f is given by

$$f(\theta) = 2 \sum_{k=0}^{\infty} \frac{\sin(k\theta) + \cos(k\theta)}{(1+k)^{1.1}}.$$

Clearly f is in the Wiener class. The spectra of A_n , $B_n^{-1}A_n$, $C_n^{-1}A_n$ and $D_n^{-1}A_n$ for $n = 32$ are represented in Figure 1. Table 1 shows the number of iterations required to make $\|r_q\|_2/\|r_0\|_2 < 10^{-7}$, where r_q is the residual

vector after q iterations. The right hand side b is the vector of all ones and the zero vector is our initial guess. We see that as n increases, the number of iterations increases like $O(\log n)$ for the original matrix A_n , while it stays almost the same for the preconditioned matrices. Moreover, all preconditioned systems converge at the same rate for large n .

n	A_n	$B_n^{-1}A_n$	$C_n^{-1}A_n$	$D_n^{-1}A_n$
16	13	7	8	7
32	15	6	7	6
64	18	7	7	7
128	19	7	7	7
256	21	7	7	7

Table 1. Number of Iterations for Different Systems

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Figure 1

Spectra of the Preconditioned Systems

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